

Statistical errors for the edm experiment

1 Introduction

Technically, the main goal of the new edm experiment* is to find nonvanishing linear slope of $R \equiv \frac{N_u - N_d}{N_u + N_d}$ as a function of time. Here N_u and N_d are number of events (decay electrons) counted by upper and lower detectors, respectively. In this paper we study statistical properties of the chi-squared fit of function $R(t)$ with linear function $f(t) = kt$. Parameter k is what we really want to find in this experiment.

2 Statistical error for R value

First of all, let us find statistical error of R for some particular time bin, t_n . Denote N_u^o and N_d^o and R_o to be the number of up and down events and R , respectively, for the "ensemble average" (or "true") values, and introduce $N \equiv N_u^o + N_d^o$ and $\Delta \equiv N_u^o - N_d^o$. We also define statistical fluctuation of R as $\delta R \equiv R - R_o$ and, similarly, statistical fluctuations $\delta N_u \equiv N_u - N_u^o$, $\delta N_d \equiv N_d - N_d^o$. Then

$$\begin{aligned} R = R_o + \delta R &= \frac{N_u - N_d}{N_u + N_d} = \frac{(N_u^o + \delta N_u) - (N_d^o + \delta N_d)}{(N_u^o + \delta N_u) + (N_d^o + \delta N_d)} = \frac{\Delta + \delta N_u - \delta N_d}{N + \delta N_u + \delta N_d} \approx \\ &\approx \frac{\Delta}{N} + \frac{\delta N_u - \delta N_d}{N} - \frac{\Delta}{N^2}(\delta N_u + \delta N_d) = R_o + \delta N_u \frac{N - \Delta}{N^2} - \delta N_d \frac{N + \Delta}{N^2} = \\ &= R_o + \delta N_u \frac{2N_d}{N^2} - \delta N_d \frac{2N_u}{N^2} \end{aligned} \quad (1)$$

thus $\delta R = \delta N_u \frac{2N_d}{N^2} - \delta N_d \frac{2N_u}{N^2}$. Then statistical error for R , σ_R , can be found as ensemble average of $(\delta R)^2$:

$$\begin{aligned} \sigma_R^2 &\equiv \langle (\delta R)^2 \rangle = \langle (\delta N_u)^2 \rangle \frac{4N_d^2}{N^4} + \langle (\delta N_d)^2 \rangle \frac{4N_u^2}{N^4} = \frac{4}{N^4} (N_u N_d^2 + N_d N_u^2) = \\ &= \frac{4N_u N_d}{N^4} (N_u + N_d) = \frac{4N_u N_d}{N^3} \end{aligned} \quad (2)$$

Thus $\sigma_R = 2\sqrt{\frac{N_u N_d}{N^3}}$ for arbitrary N_u and N_d . For our experiment, however, we would expect rather small difference (if any) between N_u and N_d . Therefore $N_u \approx N_d \approx N/2$ and $\sigma_R \approx 1/\sqrt{N}$.

*see J-PARC Letter of Intent: Search for a Permanent Muon Electric Dipole Moment at the $10^{-24} e\text{-cm}$ Level, January 9, 2003

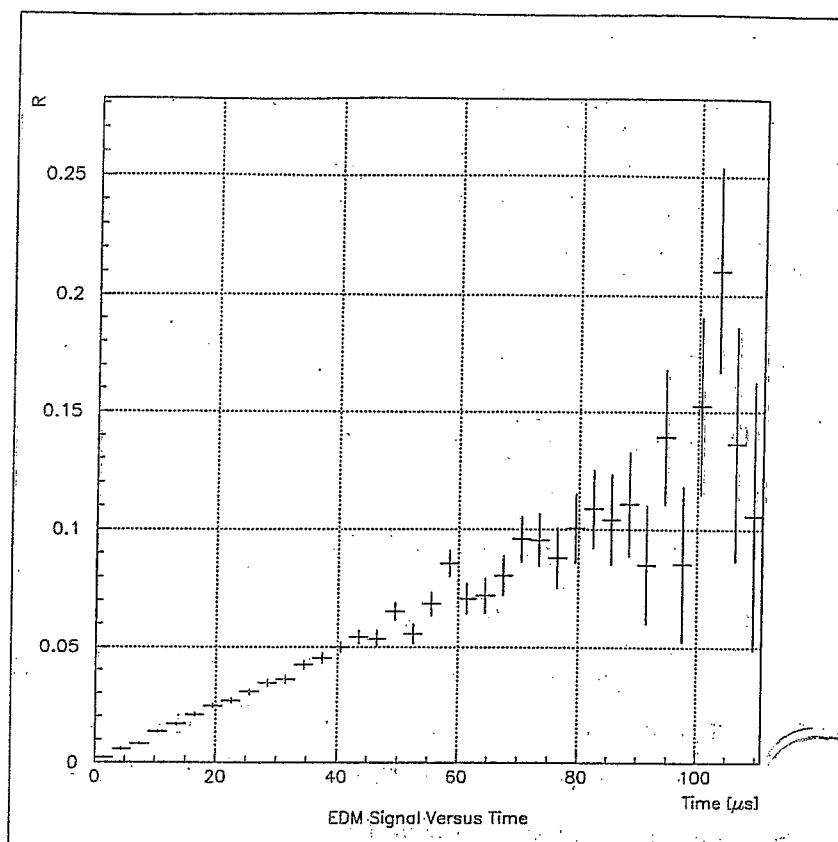


Figure 3: MC simulation of the muon EDM signal, $R = \frac{N_{up} - N_{down}}{N_{up} + N_{down}}$, versus time.

3. Selection of the pion momentum (curved solenoid section).
4. Decay of pions to muons (pion decay and muon transport section).
5. Selection of the muon momentum, and
6. Compression of the muon momentum spread by using fixed field alternating gradient (FFAG) phase rotator.

Since the momentum acceptance of the muon storage EDM ring would be limited to a level of $\Delta p/p \simeq 2\%$, it is desirable to employ the phase rotation technique to reduce the energy spread with minimum loss of muons. By using phase rotation, the momentum spread of the muon beam can be reduced by more than an order of magnitude, from $\Delta p/p = \pm 30\%$ to $\Delta p/p \simeq \pm 2\%$. This would provide a sufficient number of polarized muons injected into the muon storage EDM ring.

To accomplish a phase rotated muon beam for the EDM experiment, a new beam facility has to be constructed. We refer to it as PRISM-II. The major difference from PRISM-I (which is primarily for the $\mu^- - e^-$ conversion experiment) is in the momentum

Fitting $R(t)$

In this section we repeat our evaluations of statistical errors and correlations of fit parameters for the two parameters fit (slope plus offset) done in the previous section, but now without any assumptions on start time t_s , stop time t_m , muon lifetime τ and beam polarization lifetime τ_p . In this section we shall denote initial number of muons as N_o to match Yannis' notation, thus

$$N(t) = N_o e^{-t/\tau} \quad (20)$$

We denote a to be a constants offset of $R(t)$ as before, but for the slope k we'll use Yannis' expression $P_o \rho e^{-t/\tau} d$, where P_o is initial beam polarization, d is edm and $\rho = \frac{AE^*}{S}$, where A is asymmetry, $E^* = E_r + uB$ and S is particle's spin: $S = \hbar/2$ for muon and $S = \hbar$ for deuteron. Thus our two parameter fit function now has following form:

$$f(t) = a + (P_o \rho e^{-t/\tau_p} d) \cdot t \quad (21)$$

and we fit for a and d . In order to avoid confusions, we shall define parameters a , N_o and P_o as offset, initial number of muons and initial beam polarization, respectively, at the start time, i.e. at $t = t_s$. Then eqs.(20,21) read

$$N(t) = N_o e^{-(t-t_s)/\tau} \quad (22)$$

$$f(t) = a + (P_o \rho e^{-(t-t_s)/\tau_p} d) \cdot (t - t_s) \quad (23)$$

Equations (22) and (23) are explicitly invariant with respect to arbitrary choice of time origin. For our evaluations we shall need total number of events:

$$\begin{aligned} \mathcal{N} &= \sum_i N_i \approx \frac{1}{b} \int_{t_s}^{t_m} N(t) dt = \frac{1}{b} \int_{t_s}^{t_m} N_o e^{-(t-t_s)/\tau} dt = \frac{N_o \tau}{b} (-e^{-t_m/\tau} + e^{-t_s/\tau}) e^{t_s/\tau} = \\ &= \frac{N_o \tau}{b} (-e^{-T/\tau} + 1) \end{aligned} \quad (24)$$

where b is a width of time bin and $T \equiv t_m - t_s$. Alternatively, eq.(24) can be used to express b :

$$b = \frac{N_o \tau}{\mathcal{N}} (-e^{-T/\tau} + 1) \quad (25)$$

Find derivatives of fit function $f(t)$ with respect to its parameters a and d :

$$f'_1 \equiv \frac{\partial f}{\partial a} = 1 \quad (26)$$

$$f'_2 \equiv \frac{\partial f}{\partial d} = (P_o \rho e^{-(t-t_s)/\tau_p}) \cdot (t - t_s) \quad (27)$$

Then find elements of matrix \mathcal{A} :

$$\mathcal{A}_{ij} = \sum_n f'_i f'_j N = \frac{1}{b} \int_{t_s}^{t_m} f'_i f'_j N(t) dt \quad (28)$$

Explicitly:

$$\mathcal{A}_{11} = \frac{\mathcal{N}}{N_o \tau (-e^{-T/\tau} + 1)} \int_{t_s}^{t_m} N(t) dt = \mathcal{N} \quad (29)$$

$$\begin{aligned} \mathcal{A}_{12} &= \mathcal{A}_{21} = \frac{\mathcal{N}}{N_o \tau (-e^{-T/\tau} + 1)} \int_{t_s}^{t_m} (P_o \rho e^{-(t-t_s)/\tau_p}) (t - t_s) \times N_o e^{-(t-t_s)/\tau} dt = \\ &= \frac{\mathcal{N} P_o \rho}{\tau (-e^{-T/\tau} + 1)} \int_0^{t_m-t_s} e^{-\xi/\tau_p} e^{-\xi/\tau} \xi d\xi = \frac{\mathcal{N} P_o \rho}{\tau (-e^{-T/\tau} + 1)} \int_0^T e^{-\xi/\tau_1} \xi d\xi = \\ &= \mathcal{N} P_o \rho \frac{\tau_1}{\tau} \frac{-e^{-T/\tau_1} (T + \tau_1) + \tau_1}{-e^{-T/\tau} + 1} \end{aligned} \quad (30)$$

$$\begin{aligned} \mathcal{A}_{22} &= \frac{\mathcal{N}}{N_o \tau (-e^{-T/\tau} + 1)} \int_{t_s}^{t_m} (P_o \rho e^{-(t-t_s)/\tau_p})^2 (t - t_s)^2 \times N_o e^{-(t-t_s)/\tau} dt = \\ &= \frac{\mathcal{N} (P_o \rho)^2}{\tau (-e^{-T/\tau} + 1)} \int_0^{t_m-t_s} e^{-2\xi/\tau_p} e^{-\xi/\tau} \xi^2 d\xi = \frac{\mathcal{N} (P_o \rho)^2}{\tau (-e^{-T/\tau} + 1)} \int_0^T e^{-\xi/\tau_2} \xi^2 d\xi = \\ &= \mathcal{N} (P_o \rho)^2 \frac{\tau_2}{\tau} \frac{-e^{-T/\tau_2} (T^2 + 2T\tau_2 + 2\tau_2^2) + 2\tau_2^2}{-e^{-T/\tau} + 1} \end{aligned} \quad (31)$$

Here we use substitution $\xi = t - t_s$ in some integrals and introduce parameters τ_1 and τ_2 such that

$$\boxed{\frac{1}{\tau_1} = \frac{1}{\tau} + \frac{1}{\tau_p}} \quad \text{and} \quad \boxed{\frac{1}{\tau_2} = \frac{1}{\tau} + \frac{2}{\tau_p}} \quad (32)$$

Now we find the determinant of matrix \mathcal{A}_{ij} :

$$\begin{aligned} \mathcal{A}_{11}\mathcal{A}_{22} - \mathcal{A}_{12}\mathcal{A}_{21} &= \left[\frac{\mathcal{N} P_o \rho}{\tau (-e^{-T/\tau} + 1)} \right]^2 \times \\ &\times \left[\left(-e^{-T/\tau_2} (T^2 \tau_2 + 2T\tau_2^2 + 2\tau_2^3) + 2\tau_2^3 \right) \times \left(-e^{-T/\tau} \tau + \tau \right) - \left(-e^{-T/\tau_1} (T\tau_1 + \tau_1^2) + \tau_1^2 \right)^2 \right] \end{aligned} \quad (33)$$

and hence

$$\begin{aligned} \sigma_d^2 &= \frac{\mathcal{A}_{11}}{\mathcal{A}_{11}\mathcal{A}_{22} - \mathcal{A}_{12}\mathcal{A}_{21}} = \frac{1}{\mathcal{N}} \left[\frac{\tau (-e^{-T/\tau} + 1)}{P_o \rho} \right]^2 \times \\ &\times \left[\left(-e^{-T/\tau_2} (T^2 \tau_2 + 2T\tau_2^2 + 2\tau_2^3) + 2\tau_2^3 \right) \times \left(-e^{-T/\tau} \tau + \tau \right) - \left(-e^{-T/\tau_1} (T\tau_1 + \tau_1^2) + \tau_1^2 \right)^2 \right]^{-1} \end{aligned} \quad (34)$$

Equation (34) is valid for arbitrary values of parameters T , τ and τ_p . In the next section we consider several particular cases of relations between these parameters, for which eq.(34) may be simplified.

5.1 $T \gg \tau, \tau_p$

As it evident from eq.(34), the obvious simplification may be achieved if $T \gg \tau, \tau_p$. In such a case we have

$$\boxed{\sigma_d^2 = \frac{1}{\mathcal{N}} \left(\frac{\tau}{P_o \rho} \right)^2 \times \left[2\tau_2^3 \tau - \tau_1^4 \right]^{-1}} \quad (35)$$

which is much simpler than initial equation (34). In turn, eq.(35) may be simplified more if $\tau_p \gg \tau$ (thus $\tau_1, \tau_2 \approx \tau$):

$$\boxed{\sigma_d^2 = \frac{1}{\mathcal{N} (P_o \rho)^2 \tau^2}} \quad (36)$$

and if $\tau \gg \tau_p$ (thus $\tau_1 \approx \tau_p, \tau_2 \approx \frac{1}{2}\tau_p$):

$$\boxed{\sigma_d^2 = \frac{1}{\mathcal{N} (P_o \rho)^2} \frac{4\tau}{\tau_p^3}} \quad (37)$$

We note that eq.(36) is in agreement with eq.(18) in section 2.

5.2 $\tau_p \gg \tau$

For this case $\tau_1, \tau_2 \approx \tau$ and

$$\begin{aligned} \sigma_d^2 &= \frac{1}{\mathcal{N}} \left[\frac{\tau (-e^{-T/\tau} + 1)}{P_o \rho} \right]^2 \times \\ &\times \left[(-e^{-T/\tau} (T^2 \tau + 2T\tau^2 + 2\tau^3) + 2\tau^3) \times (-e^{-T/\tau} \tau + \tau) - (-e^{-T/\tau} (T\tau + \tau^2) + \tau^2)^2 \right]^{-1} = \\ &= \frac{1}{\mathcal{N}} \left[\frac{\tau (-e^{-T/\tau} + 1)}{P_o \rho} \right]^2 \times [e^{-2T/\tau} \tau^4 - e^{-T/\tau} (T^2 \tau^2 + 2\tau^4) + \tau^4]^{-1} = \\ &= \frac{1}{\mathcal{N}} \left[\frac{\tau (-e^{-T/\tau} + 1)}{P_o \rho} \right]^2 \times [\tau^4 (e^{-T/\tau} - 1)^2 - e^{-T/\tau} T^2 \tau^2]^{-1} = \\ &= \frac{1}{\mathcal{N} (P_o \rho)^2} \times \left[\tau^2 - \frac{e^{-T/\tau} T^2}{(e^{-T/\tau} - 1)^2} \right]^{-1} \end{aligned} \quad (38)$$

This equation can be simplified more for particular values of parameters T :

- for $\tau_p \gg \tau \gg T$

$$\boxed{\sigma_d^2 = \frac{1}{\mathcal{N} (P_o \rho)^2} \frac{12}{T^2}} \quad (39)$$

- for $\tau_p \gg T \gg \tau$

$$\boxed{\sigma_d^2 = \frac{1}{\mathcal{N} (P_o \rho)^2 \tau^2}} \quad (40)$$

- the last possible case in this section, $T \gg \tau_p \gg \tau$, was already considered in the previous section, see eq.(36).

5.3 $\tau \gg \tau_p$

For this case $\tau_1 \approx \tau_p$ and $\tau_2 \approx \frac{1}{2}\tau_p$. Substitution of these in eq.(34), though, doesn't simplify the latter much. Therefore here we consider more specific cases only:

- for $\tau \gg \tau_p \gg T$

$$\boxed{\sigma_d^2 = \frac{1}{\mathcal{N}} \left[\frac{T}{P_o \rho} \right]^2 \times \left[\frac{1}{12} T^4 \right]^{-1} = \frac{1}{\mathcal{N} (P_o \rho)^2} \frac{12}{T^2}} \quad (41)$$

which is, quite predictably, same as in eq.(39);

- for $\tau \gg T \gg \tau_p$

$$\boxed{\sigma_d^2 = \frac{1}{\mathcal{N}} \left[\frac{T}{P_o \rho} \right]^2 \times \left[\frac{1}{4} \tau_p^3 T - \tau_p^4 \right]^{-1} \approx \frac{1}{\mathcal{N}} \left[\frac{T}{P_o \rho} \right]^2 \times \left[\frac{1}{4} \tau_p^3 T \right]^{-1} = \frac{1}{\mathcal{N} (P_o \rho)^2} \frac{4T}{\tau_p^3}} \quad (42)$$

- the last possible case in this section, $T \gg \tau \gg \tau_p$, was already considered in section 5.1, see eq.(37). Note that eqs.(37) and (42) are completely identical except for exchange $T \leftrightarrow \tau$.

Deuteron edm storage ring simulations

Central orbit issues

Continuous mg. field: $B = 0.13\text{T}$ $P = 1.07\text{GeV}$

Piece-wise m. field: $B = 0.13\text{T}$ $P = 0.74\text{GeV}$

Introduce E-field (g-2 compensation):

$B = 0.13\text{T}$ $E = 1.95\text{MV/m}$ $P = 0.63\text{GeV}$

$B = 0.174\text{T}$ $E = 3.5\text{MV/m}$ $P = 0.83\text{GeV}$

Beam storage issues

~~D~~ More realistic E-field

Quadrupoles

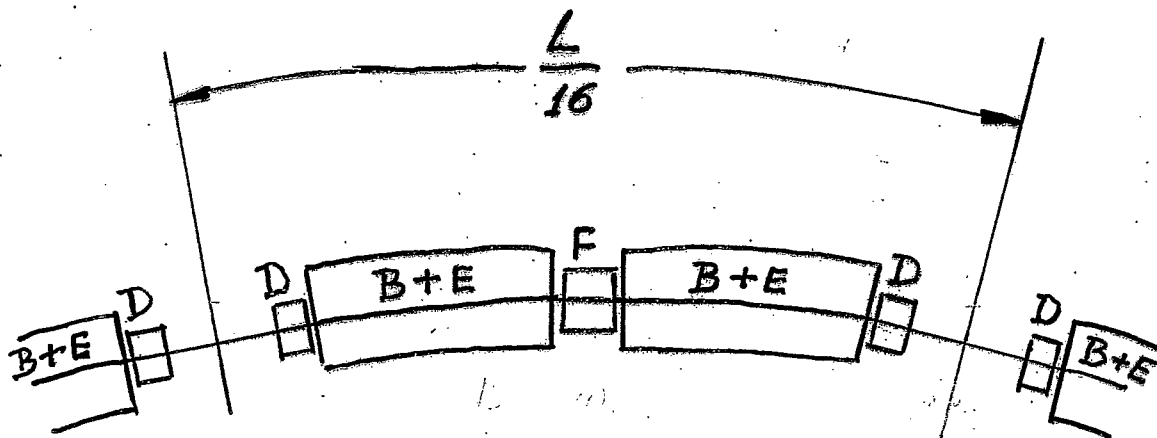
3-dimensional fields (opera?)

RF cavities

Field perturbations (imperfections, misalign...)

Others (see Yuri's report)

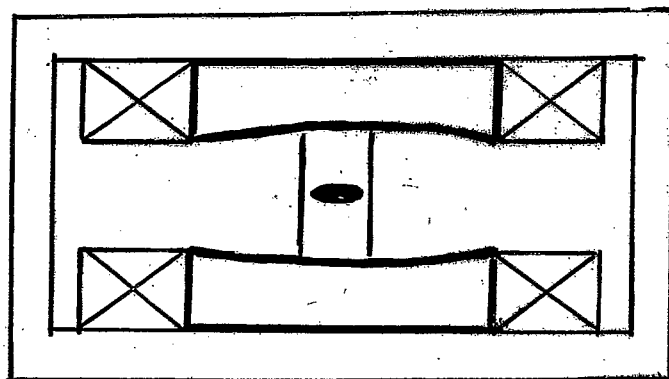
Y. Orlov



$$L/16 \sim 12 \text{ m}$$

$$l_{BE} \sim 4.5 \text{ m}$$

$$\langle R \rangle \sim 30 \text{ m}$$



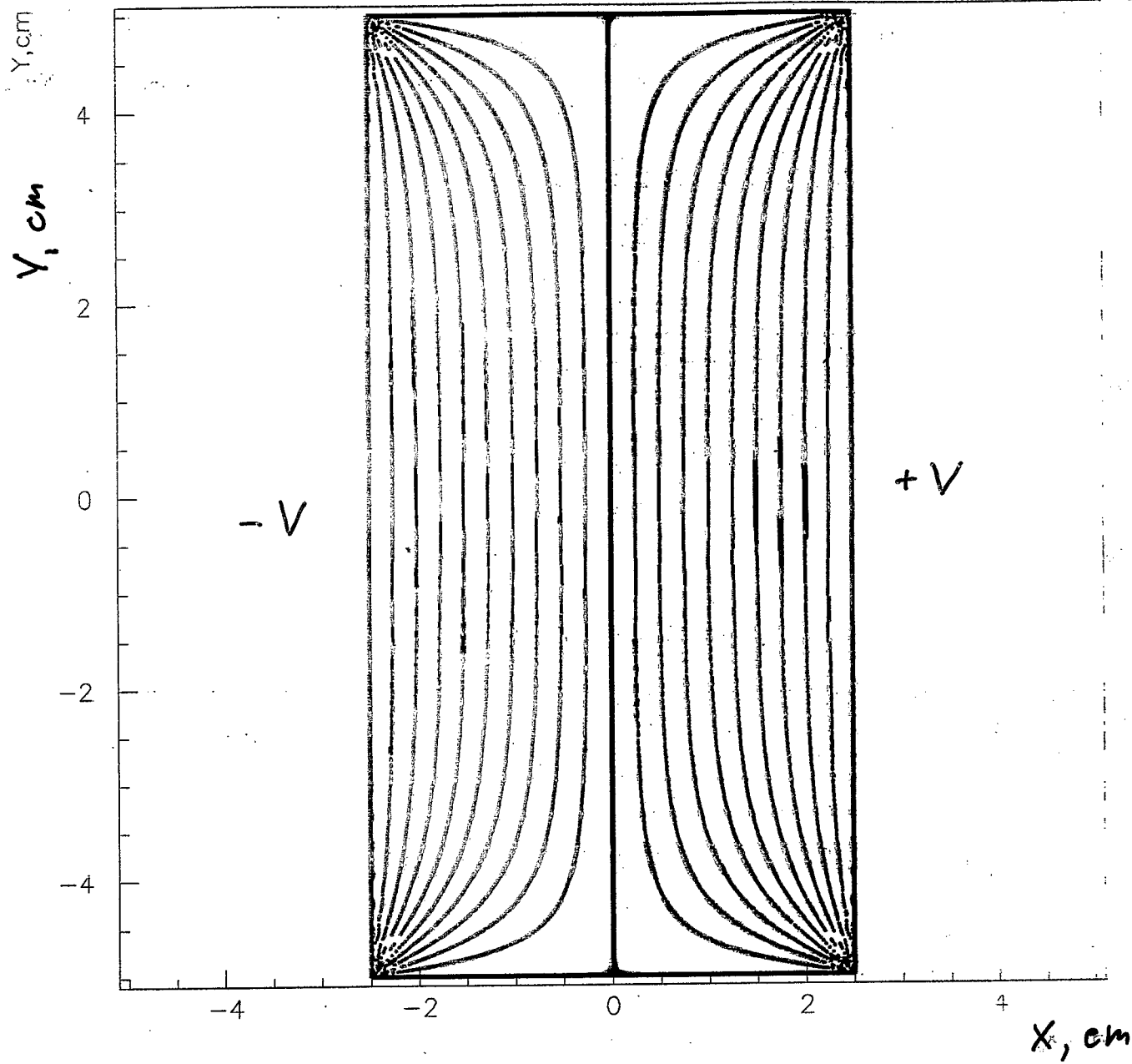
~12"

~20"

$$E \sim 3.5 \text{ MV/m}$$

$$B \sim 0.13 \text{ T}$$

$$\left(\frac{\partial^2 B / \partial x^2}{2B} \left(\frac{x}{R} \right)^2 \sim 10^{-4} - 10^{-3} \right)$$



Equating right hand sides of eqs.(10) and (13) and redenote index m back to n we obtain equation for A_n :

$$A_n = \frac{2V}{\pi} \left(\frac{1}{n + \frac{1}{2}} \right) \left(\frac{(-1)^n}{\sinh \left[\left(n + \frac{1}{2} \right) \frac{\pi a}{b} \right]} \right) \quad (14)$$

and hence solution for the electrostatic potential is

$$\phi(x, y) = \frac{2V}{\pi} \sum_{n=0}^{\infty} \left(\frac{1}{n + \frac{1}{2}} \right) \left(\frac{(-1)^n}{\sinh \left[\left(n + \frac{1}{2} \right) \frac{\pi a}{b} \right]} \right) \times \sinh \left[\left(n + \frac{1}{2} \right) \frac{\pi x}{b} \right] \times \cos \left[\left(n + \frac{1}{2} \right) \frac{\pi y}{b} \right] \quad (15)$$

2 Electrostatic field

From electrostatic potential given in eq.(15) find electrostatic fields:

$$E_x = -\frac{\partial \phi(x, y)}{\partial x} = -\frac{2}{b} V \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{\sinh \left[\left(n + \frac{1}{2} \right) \frac{\pi a}{b} \right]} \right) \times \cosh \left[\left(n + \frac{1}{2} \right) \frac{\pi x}{b} \right] \times \cos \left[\left(n + \frac{1}{2} \right) \frac{\pi y}{b} \right] \quad (16)$$

$$E_y = -\frac{\partial \phi(x, y)}{\partial y} = \frac{2}{b} V \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{\sinh \left[\left(n + \frac{1}{2} \right) \frac{\pi a}{b} \right]} \right) \times \sinh \left[\left(n + \frac{1}{2} \right) \frac{\pi x}{b} \right] \times \sin \left[\left(n + \frac{1}{2} \right) \frac{\pi y}{b} \right] \quad (17)$$

Since we are interested in electrical field in some region at and around of deuteron central orbit, it is convenient to represent electrical field in terms of multipoles.

First, we use hyperbolic-trigonometric equation

$$\cos(a + ib) = \cos a \cos ib - \sin a \sin ib = \cos a \cosh b - i \sin a \sinh b \quad (18)$$

and rewrite eqs.(16) and (17) as

$$E_x + iE_y = -\frac{2}{b} V \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{\sinh \left[\left(n + \frac{1}{2} \right) \frac{\pi a}{b} \right]} \right) \times \cos \left[\left(n + \frac{1}{2} \right) \frac{\pi}{b} (x + iy) \right] \quad (19)$$

$$\text{or } E_x + iE_y = -\frac{2}{b} V \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{\sinh \left[\left(n + \frac{1}{2} \right) \frac{\pi a}{b} \right]} \right) \times \cos \left[\left(n + \frac{1}{2} \right) \frac{\pi}{b} Z \right] \quad (20)$$

where $Z = x + iy$. Elements of decomposition of cosine in the right hand side of eq.(20) in powers of Z gives multipoles value of electric field:

$$\begin{aligned} E_x + iE_y &= -\frac{2}{b} V \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{\sinh \left[\left(n + \frac{1}{2} \right) \frac{\pi a}{b} \right]} \right) \times \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left[\left(n + \frac{1}{2} \right) \frac{\pi}{b} \right]^{2m} Z^{2m} = \\ &= -\frac{2}{b} V \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(\frac{\pi Z}{b} \right)^{2m} \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right)^{2m} \frac{(-1)^n}{\sinh \left[\left(n + \frac{1}{2} \right) \frac{\pi a}{b} \right]} = \\ &= V \sum_{m=0}^{\infty} C_m Z^{2m} \end{aligned} \quad (21)$$

For our case, numerical calculation for the coefficients C_n gives:

$$C_0 = -0.39701767 \text{ [cm}^{-1}\text{]} \quad (22)$$

$$C_1 = 2.34160101 \cdot 10^{-3} \text{ [cm}^{-1}\text{]} \quad (23)$$

$$C_2 = 3.01262788 \cdot 10^{-4} \text{ [cm}^{-3}\text{]} \quad (24)$$

$$C_3 = 1.44295465 \cdot 10^{-5} \text{ [cm}^{-5}\text{]} \quad (25)$$

$$C_4 = 2.50862322 \cdot 10^{-7} \text{ [cm}^{-7}\text{]} \quad (26)$$

$$C_5 = -5.76732932 \cdot 10^{-9} \text{ [cm}^{-9}\text{]} \quad (27)$$

$$C_6 = -4.74165277 \cdot 10^{-10} \text{ [cm}^{-11}\text{]} \quad (28)$$

3 Electric field in the deuteron storage ring

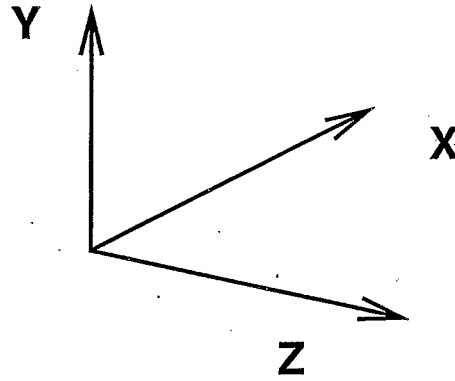


Figure 2: Geometry of deuteron storage ring

System of coordinates for the deuteron storage ring is shown in Fig.2. Axis X is pointed radially outward, axis Y is vertically upward and Z is azimuthally along trajectory of deuterons. In such a system magnetic field \vec{B} is vertically upward and electrical field \vec{E} required to compensate $g - 2$ precession of deuteron's spin is radially outward. Therefore for this case we must assume that voltage V as given in Fig.1 has negative value. For our deuteron storage ring studies we have chosen [dipole] electric field $3.5 \text{ MV/m} = 0.035 \text{ MV/cm}$. Then from eqs.(21) and (28) we can find required voltage V on electrodes:

$$(E_x + iE_y)_{\text{dipole}} = V C_0 = -0.39701767 \text{ cm}^{-1} V \text{ [V]}$$

hence $E_y \text{ dipole} = 0$

and $E_x \text{ dipole} = -0.39701767 V \text{ [V/cm]} \quad (29)$

and therefore $\underline{V \text{ [V]}} = \frac{0.035 \text{ MV/cm}}{-0.39701767 \text{ cm}^{-1}} = -0.08816 \text{ MV} = \underline{-88.16 \text{ kV}}$